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CONTRIBUTIONS FROM THE JEFFERSON PHYSICAL LABORATORY, HARVARD UNIVERSITY.

THE CONCEPTION OF THE DERIVATIVE OF A SCALAR POINT FUNCTION WITH RESPECT TO ANOTHER SIMILAR FUNCTION.

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In modern treatises on Mathematical Physics it is customary to define the derivative of a scalar function, taken at a given point in space in a given direction, in a manner which emphasizes the fact that this derivative is an invariant of a transformation of coördinates. According to this definition,[1] if through the point P a straight line be drawn in a fixed direction (s), if on this line a point P' be taken near P so that PP' has the direction s, and if u_P, $u_{P'}$ be used to represent the values at these points of the scalar point function u, then if the ratio

$$\frac{u_{P'} - u_P}{PP'} \tag{1}$$

approaches a limit as P' approaches P, this limit is called the derivative of u, at P, in the direction s. If u happens to be defined in terms of a system of orthogonal Cartesian coördinates, x, y, z, and has continuous derivatives with respect to these coördinates everywhere within a certain region, the limit just mentioned exists in this region and its value is

$$\frac{\partial u}{\partial x} \cdot \cos(x, s) + \frac{\partial u}{\partial y} \cdot \cos(y, s) + \frac{\partial u}{\partial z} \cdot \cos(z, s). \tag{2}$$

[1] Hamilton, Elements of the Theory of Quaternions; Tait, Elementary Treatise on Quaternions; Gibbs, Vector Analysis; Maxwell, Treatise on Electricity and Magnetism; Webster, Dynamics of Particles and of Rigid, Elastic, and Fluid Bodies; Jeans, Mathematical Theory of Electricity and Magnetism; Lamé, Leçons sur les Coordonnées Curvilignes; Peirce, Theory of the Newtonian Potential Function; Generalized Space Differentiation of the Second Order; Czuber, Wienerberichte, **101**$_A$, 1417 (1892); Boussinesq, Cours d'Analyse Infinitésimale; H. Weber, Die Partiellen Differential-Gleichungen der Mathematischen Physik.

Of all the numerical values which the derivative of u can have at a given point, the greatest is to be found by making s normal to the level surface of u which passes through the point. This maximum value,

$$\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right]^{\frac{1}{2}} \tag{3}$$

is usually regarded as the value at the point of a vector point function called the gradient vector of u, the lines of which cut orthogonally the level surfaces of u, and the components of which parallel to the coördinate axes are

$$\frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y},\ \frac{\partial u}{\partial z}. \tag{4}$$

This vector is, of course, lamellar.

The value of the tensor of the gradient vector is often called simply the "gradient" of u and is denoted by h_u. If at any point a straight line be drawn in the direction (n) normal to the level surface of u, in the sense in which u increases, and if a length h_u be laid off on this line, the projection,

$$h_u \cdot \cos(n, s), \tag{5}$$

of this length on any other direction (s) is numerically equal to the derivative of u in the direction s.

Most physical quantities — such as temperature, barometric pressure, density, inductivity — present themselves to the investigator as single valued point functions, which, except perhaps at one or more given surfaces of discontinuity, are differentiable in the sense just considered.

It is often desirable to differentiate a scalar function, u, at a point, in the direction in which another scalar function, v, increases fastest, and if (u, v) represents the angle between the gradient vectors of u and v at the point, the derivative is evidently equal to

$$h_u \cdot \cos(u, v). \tag{6}$$

It frequently happens that in a question of maxima and minima, one wishes to determine the greatest (or the smallest) value which a quantity U may have, subject to the condition that another quantity V shall have a given value (V_o). If these quantities can be represented by point functions, the problem geometrically considered requires one to find the parameter of a surface of the constant U family, which is tangent to the surface of the V family upon which V is everywhere

equal to V_o; but at the point of tangency, the derivative of the function U in any direction in the tangent plane of the V surface is zero, that is, the normals to the U and V surfaces coincide, so that

$$\frac{\frac{\partial u}{\partial x}}{\frac{\partial v}{\partial x}} = \frac{\frac{\partial u}{\partial y}}{\frac{\partial v}{\partial y}} = \frac{\frac{\partial u}{\partial z}}{\frac{\partial v}{\partial z}}, \tag{7}$$

and these familiar equations usually furnish some general information about the problem independent of the value of V_o. As an extremely

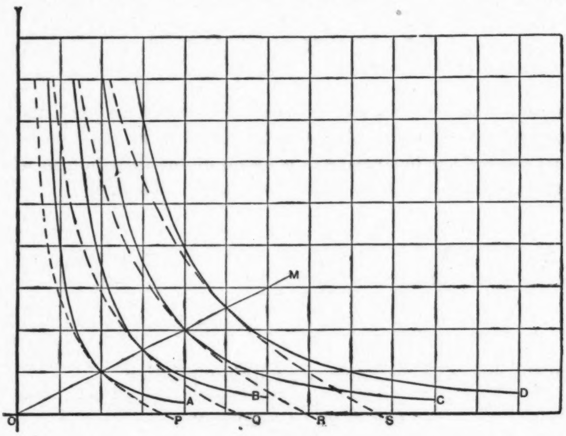

FIGURE 1.

simple example we may take the familiar problem concerning the relative dimensions of an open tank of square base ($x \times x$) and height y, which shall hold a given quantity ($V = x^2 . y$) of water and have the smallest wet surface ($U = x^2 + 4xy$). Here we have the curve D of the V family, which has the given parameter, V_o, and are required to find that member of the P, Q, R, S family which touches D. The equation (7) becomes in this case $2y = x$, and it appears (Figure 1) that the curves of the two families which pass through any point of the line OM are at that point tangent to each other.

It is sometimes necessary to differentiate a point function, u, at a point P, in the direction of the line through the point, along which

two other point functions, v, w, are constant; that is, along the line $v = v_P$, $w = w_P$. If

$$L = \begin{vmatrix} \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}, \quad M = \begin{vmatrix} \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \\ \frac{\partial w}{\partial z} & \frac{\partial w}{\partial x} \end{vmatrix}, \quad N = \begin{vmatrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} \tag{8}$$

and if $R^2 = L^2 + M^2 + N^2$ — which is equal to $h_v^2 \cdot h_w^2$, if v and w are orthogonal — this direction is defined by the cosines L/R, M/R, N/R, and the derivative required is

$$\frac{1}{R}\left(L \cdot \frac{\partial u}{\partial x} + M \cdot \frac{\partial u}{\partial y} + N \cdot \frac{\partial u}{\partial z}\right). \tag{9}$$

If the maxima and minima of the function $u = f(x, y, z)$ are to be found under the condition that the functions v, w shall have given numerical values, the derivative of u taken in the direction in which v and w are constant must be made to vanish. Thus, if

$$u = x^2 + y^2 + z^2,$$

and if the conditions are

$$xyz = c^3 \text{ and } x + y = d,$$

equation (9) yields immediately the required relation

$$(xy + z^2)(y - x) = 0.$$

When $f'(u)$ is positive, the direction of the gradient vector of $f(u)$ coincides with that of the gradient vector of u itself: these directions are opposed when $f'(u)$ is negative. The tensors of both vectors are always positive. If

$$w = f(u), \quad h_w^2 = [f'(u)]^2 \cdot h_u^2, \text{ and } \cos(w, s) = \cos(u, s):$$

in particular, when

$$w = 1/u, \quad h_w = h_u/u^2 \text{ and } \cos(w, s) = -\cos(u, s),$$

so that

$$\frac{\partial}{\partial s}\left(\frac{1}{u}\right) = -\frac{\partial u}{\partial s} \cdot \frac{1}{u^2}$$

If u is the distance (r) to a point on a curve (s) from a fixed point outside the curve,

$$\frac{\partial r}{\partial s} = + \cos (s, r), \ \frac{\partial}{\partial s}\left(\frac{1}{r}\right) = - \frac{\cos (s, r)}{r^2}.$$

Any function of the complex variable $(ax + by + iz\sqrt{a^2 + b^2})$ has a gradient identically equal to zero, but every differentiable real point function has a gradient in general different from zero. The gradient of a function may be constant throughout a region of space: if the gradient of u is constant, the surfaces upon each of which u is constant form a parallel system. If the gradient of a function, u, is either constant or expressible in terms of u, any differentiable function of u has a gradient either constant or expressible in terms of u. If the gradient of u is expressible in terms of u alone $[h_u = f(u)]$, it is possible to form a function, $a\int \frac{du}{f(u)}$, of u the gradient of which shall be constant. If h_u is neither constant nor expressible in terms of u, no function of u exists the gradient of which is expressible in terms of u. The functions $u = \sin (x + y + z)$, $v = \sin (x + 2y - 3z)$, $w = \sin (5x - 4y - z)$ illustrate the fact that the gradient of each of three orthogonal point functions may be expressible in terms of the function itself.

If the gradient of each of two orthogonal point functions, u, v, were expressible as the product of a function of u and a function of v, so that $h_u = U_1 \cdot V_1$, and $h_v = U_2 \cdot V_2$, it would be possible to form two functions $\left[\int \frac{du}{U_1}, \int \frac{dv}{V_2}\right]$ of u alone and of v alone, respectively, the gradient of each of which would be expressible in terms of the other. If the gradient vectors of two functions have the same direction at every point of space, one of these functions is expressible in terms of the other. If the gradients of two real functions, u, v, are everywhere equal while the directions of their gradient vectors are different,

$$\frac{\partial(u+v)}{\partial x}\cdot\frac{\partial(u-v)}{\partial x} + \frac{\partial(u+v)}{\partial y}\cdot\frac{\partial(u-v)}{\partial y} + \frac{\partial(u+v)}{\partial z}\cdot\frac{\partial(u-v)}{\partial z} = 0, \quad (10)$$

and the functions $[u + v]$, $[u - v]$ are orthogonal, as are $F(u + v)$, $f(u - v)$, where F and f are any differentiable functions. If u and v are orthogonal functions, the functions $[F(u) + f(v)]$, $[F(u) - f(v)]$ have gradients numerically equal to each other at every point.

Two scalar point functions, the level surfaces of which are neither coincident nor orthogonal, may have gradients each of which is ex-

pressible in terms of the other: the gradient of $v = \frac{4}{3}x^3 - 4xy^2$ is equal at every point of the xy plane to the square of the gradient of $u = x^2 - y^2$. If u and v are orthogonal functions of x and y, the product of their gradients is equal to the Jacobian,

$$\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}.$$

The differential equation

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = k^2,$$

which leads to systems of parallel surfaces, is of standard form. Its complete integral is

$$u = ax + by + z\sqrt{k^2 - a^2 - b^2} + d,$$

where a, b, d are arbitrary constants, and from this the general integral may be obtained in the usual manner.

If a direction s be determined at every point of a given region, T, by some law, the derivative of the function u becomes itself a scalar point function in T, and if this is differentiable, it may be differentiated at any point in any direction, say s. It is usually convenient to define s by means of three scalar point functions, l, m, n, the sum of the squares of which is identically equal to unity, and which represent the direction cosines of s. In this connection it is well to notice that if s has the direction at P of the tangent of a continuous curve which passes through the point, if P' be a point near P on the tangent and P'' a point near P on the curve, and if U is any differentiable scalar point function,

$$\frac{U_{P''} - U_P}{PP''}, \qquad \frac{U_{P'} - U_P}{PP'}$$

have the same limit, as P' and P'' approach P, that which has been defined as the derivative of U at P in the direction s. If, then, $\partial u/\partial s$ is differentiable

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial s}\right) = \frac{\partial}{\partial x}\left(l \cdot \frac{\partial u}{\partial x} + m \cdot \frac{\partial u}{\partial y} + n \cdot \frac{\partial u}{\partial z}\right)$$
$$= l \cdot \frac{\partial^2 u}{\partial x^2} + m \cdot \frac{\partial^2 u}{\partial x \cdot \partial y} + n \cdot \frac{\partial^2 u}{\partial x \cdot \partial z} + \frac{\partial u}{\partial x} \cdot \frac{\partial l}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial m}{\partial x} + \frac{\partial u}{\partial z} \cdot \frac{\partial n}{\partial x}, \tag{11}$$

and

$$\frac{\partial^2 u}{\partial s^2} = l^2 \cdot \frac{\partial^2 u}{\partial x^2} + m^2 \cdot \frac{\partial^2 u}{\partial y^2} + n^2 \cdot \frac{\partial^2 u}{\partial z^2} + 2lm \cdot \frac{\partial^2 u}{\partial x \cdot \partial y} + 2mn \cdot \frac{\partial^2 u}{\partial y \cdot \partial z} + 2ln \cdot \frac{\partial^2 u}{\partial x \cdot \partial z}$$
$$+ \frac{\partial u}{\partial x}\left(l \cdot \frac{\partial l}{\partial x} + m \cdot \frac{\partial l}{\partial y} + n \cdot \frac{\partial l}{\partial z}\right) + \frac{\partial u}{\partial y}\left(l \cdot \frac{\partial m}{\partial x} + m \cdot \frac{\partial m}{\partial y} + n \cdot \frac{\partial m}{\partial z}\right)$$
$$+ \frac{\partial u}{\partial z}\left(l \cdot \frac{\partial n}{\partial x} + m \cdot \frac{\partial n}{\partial y} + n \cdot \frac{\partial n}{\partial z}\right). \tag{12}$$

If s' is a direction defined by the cosines l', m', n',

$$\frac{\partial^2 u}{\partial s' \cdot \partial s} = ll' \cdot \frac{\partial^2 u}{\partial x^2} + mm' \cdot \frac{\partial^2 u}{\partial y^2} + nn' \cdot \frac{\partial^2 u}{\partial z^2}$$
$$+ (lm' + l'm)\frac{\partial^2 u}{\partial x \cdot \partial y} + (mn' + m'n)\frac{\partial^2 u}{\partial y \cdot \partial z} + (nl' + n'l)\frac{\partial^2 u}{\partial z \cdot \partial x}$$
$$+ \frac{\partial u}{\partial x}\left(l' \cdot \frac{\partial l}{\partial x} + m' \cdot \frac{\partial l}{\partial y} + n' \cdot \frac{\partial l}{\partial z}\right) + \frac{\partial u}{\partial y}\left(l' \cdot \frac{\partial m}{\partial x} + m' \cdot \frac{\partial m}{\partial y} + n' \cdot \frac{\partial m}{\partial z}\right)$$
$$+ \frac{\partial u}{\partial z}\left(l' \cdot \frac{\partial n}{\partial x} + m' \cdot \frac{\partial n}{\partial y} + n' \cdot \frac{\partial n}{\partial z}\right), \tag{13}$$

and it is clear that the order of differentiation is usually not commutative. Derivatives of this kind are often found in differential equations of orders higher than the first which define functions in terms of simple curvilinear coördinates.

If for instance spherical coördinates are to be used, the second derivative of u taken in the direction in which θ increases fastest is

$$\frac{\partial^2 u}{\partial x^2} \cdot \cos^2\theta \cos^2\phi + \frac{\partial^2 u}{\partial y^2} \cdot \cos^2\theta \sin^2 y + \frac{\partial^2 u}{\partial z^2} \cdot \sin^2\theta + \frac{2\,\partial^2 u}{\partial x \cdot \partial y} \cdot \cos^2\theta \sin\phi \cos\phi$$
$$- \frac{2\,\partial^2 u}{\partial x \cdot \partial z} \cdot \sin\theta \cos\theta \cos\phi - \frac{2\,\partial^2 u}{\partial y \cdot \partial z} \cdot \sin\theta \cos\theta \sin\phi - \frac{\partial u}{r \cdot \partial x} \cdot \sin\theta \cos\phi$$
$$- \frac{\partial u}{r \cdot \partial y} \cdot \sin\theta \sin\phi - \frac{\partial u}{r \cdot \partial z} \cdot \cos\theta \tag{14}$$

and this, which contains derivatives of the first order, is in sharp contrast to the second derivative of u taken in the direction r, which is,

$$\frac{\partial^2 u}{\partial x^2} \cdot \sin^2\theta \cos^2\phi + \frac{\partial^2 u}{\partial y^2} \cdot \sin^2\theta \sin^2\phi + \frac{\partial^2 u}{\partial z^2} \cdot \cos^2\theta + \frac{2\,\partial^2 u}{\partial x \cdot \partial y} \cdot \sin^2\theta \sin\phi \cos\phi$$
$$+ \frac{2\,\partial^2 u}{\partial y \cdot \partial z} \cdot \sin\theta \cos\theta \sin\phi + \frac{2\,\partial^2 u}{\partial z \cdot \partial x} \cdot \sin\theta \cos\theta \cos\phi. \tag{15}$$

Sometimes s and s' are fixed directions so that l, m, n, l', m', n', are constants throughout T, and in this case the coefficients of $\partial u/\partial x$, $\partial u/\partial y$, $\partial u/\partial z$ in (12) and (13) vanish. The mutual potential energy W, of two magnetic elements, M, M', of moments, m, m', can be written in the form

$$m \cdot m' \frac{\partial^2}{\partial s \cdot \partial s'}\left(\frac{1}{r}\right), \tag{16}$$

where r is the distance MM' and s, s' are the directions of the axes of the elements. The force (due to the second magnet) which tends to move the first magnet in the direction of its own axis is then

$$- m \cdot m' \frac{\partial^3}{\partial s \cdot \partial s \cdot \partial s'}\left(\frac{1}{r}\right) \tag{17}$$

and these differentiations assume that the direction cosines of s and s' are constants.

In general, if s is the direction perpendicular to the level surface of u, and if h is the scalar point function which gives the value of $\partial u/\partial s$,

$$\frac{\partial^2 u}{\partial s^2} = \left(\frac{\partial h}{\partial x}\cdot\frac{\partial u}{\partial x} + \frac{\partial h}{\partial y}\cdot\frac{\partial u}{\partial y} + \frac{\partial h}{\partial z}\cdot\frac{\partial u}{\partial z}\right)\Big/ h. \tag{18}$$

In the case of oblique Cartesian coördinates in a plane, x increases fastest in a direction which is not perpendicular to the line along which it is constant. If the angle between the coördinate axes is ω,

$$\frac{\partial u}{\partial x} = h_u \cdot \cos(x, h_u), \quad \frac{\partial u}{\partial y} = h_u \cdot \cos(y, h_u), \quad \frac{\partial u}{\partial s} = h_u \cdot \cos(s, h_u),$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\sin(y, s)}{\sin\omega} + \frac{\partial u}{\partial y}\cdot\frac{\sin(x, s)}{\sin\omega} \tag{19}$$

It is frequently necessary to differentiate one point function, U, with respect to another, u, and the process usually appears in the form of a kind of partial differentiation. If, for instance, U is to satisfy a differential equation in terms of a set of orthogonal curvilinear coördinates of which u is one, the derivatives of U with respect to u are to be taken on the assumption that the other coördinates remain constant. This large subject has been treated exhaustively in the many works on

orthogonal coördinates which have been published since Lamé's classical treatise [2] appeared.

Given a function, u, it is, however, not generally possible to find a system of orthogonal functions of which u shall be one, and it is often convenient for a physicist to differentiate a physical function, U, with respect to another, u, without considering the existence of any other related functions. A physical point function has a value at every point in space which is not altered by changing the system of coördinates which fix the position of the point, and it is well to define the derivative of U with regard to u in a manner which shall emphasize the fact that the derivative is an invariant of a change of coördinates and which shall not assume that two functions (v, w) can be found orthogonal to each other and to u. When U and u are considered by themselves and not regarded as coördinated of necessity with other similar quantities, it is usually, if not always, the case that a "normal" derivative [3] is required.

The *normal* derivative, at any point, P, of the differentiable scalar point function U, with respect to the differentiable scalar point function u, may be defined as the limit, when PP' approaches zero, of the ratio

$$\frac{U_{P'} - U_P}{u_{P'} - u_P}, \tag{20}$$

where P' is a point so chosen on the normal at P of the surface of constant u which passes through P, that $u_{P'} - u_P$ shall be positive. If (U, u) denotes the angle between the directions in which U and u increase most rapidly, the normal derivatives of U with respect to u and of u with respect to U may be written

$$\frac{h_U}{h_u} \cdot \cos(U, u) \quad \text{and} \quad \frac{h_u}{h_U} \cdot \cos(U, u). \tag{21}$$

If $h_U = h_u$ these derivatives are equal. An example of this is the equality of $\partial n / \partial r$ and $\partial r / \partial n$ in a familiar application of Green's Theorem, where n and r represent the normal distance from a given surface and the distance from a given fixed point respectively. If U and u happen to be expressed in terms of a set (x, y, z) of orthogonal

[2] Lamé, Leçons sur les Coordonnées Curvilignes et leur Diverses Applications; Salvert, Mémoire sur l'Emploi des Coordonnées Curvilignes; Darboux, Leçons sur les Systèmes Orthogonaux et les Coordonnées Curvilignes; Goursat, Cours d'Analyse Mathématique.

[3] Peirce, Short Table of Integrals, Theory of the Newtonian Potential Function; Generalized Space Differentiation of the Second Order.

Cartesian coördinates, the normal derivative of U with respect to u can be written

$$D_u U = \frac{\frac{\partial U}{\partial x}\cdot\frac{\partial u}{\partial x} + \frac{\partial U}{\partial y}\cdot\frac{\partial u}{\partial y} + \frac{\partial U}{\partial z}\cdot\frac{\partial u}{\partial z}}{h_u^2}, \tag{22}$$

and it is easy to see that this is equal to the ratio of the derivatives of U and u taken in the direction in which u increases most rapidly.

It is occasionally instructive to use the conception of normal differentiation in studying some of the general equations of Physics: thus in the uncharged dielectric about an electric distribution, the potential function, V, is connected with the inductivity of the medium, μ, by the familiar equation

$$\frac{\partial}{\partial x}\left(\mu\cdot\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\cdot\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu\cdot\frac{\partial V}{\partial z}\right) = 0, \tag{23}$$

in which μ is to be regarded as a point function discontinuous in general at each of a given set of surfaces at every point of which an equation of the form

$$\mu_1\cdot\frac{\partial V}{\partial n_1} + \mu_2\cdot\frac{\partial V}{\partial n_2} = 0 \tag{24}$$

is satisfied. Now (23) may be put into the form

$$\frac{\partial \log \mu}{\partial V} + \frac{\nabla^2 V}{h_V^2} = 0, \tag{25}$$

and according to Lamé's condition, the second term is a function of V only, if the level surfaces of V are possible level surfaces of a harmonic function.

It is easy to make from (25), by inspection, such simple deductions as those which follow in this paragraph. If V is harmonic, either the dielectric is made up of homogeneous portions separated from one another by equipotential surfaces, or the level surfaces of μ and of V are everywhere perpendicular to each other. If V, though not harmonic, satisfies Lamé's condition $[\nabla^2(V)/h_v^2 = F(V)]$ the level surfaces of the inductivity are equipotential; and if the level surfaces of V and μ are identical, V satisfies Lamé's condition. If when the plates of a condenser are kept at given potentials, the level surfaces of the inductivity of the dielectric are equipotential, the value of the potential function in

the dielectric would be unchanged if μ were changed to $\Omega.\mu$, where Ω is any scalar point function orthogonal to V. If the continuous dielectric of a condenser in which the level surfaces of the inductivity, μ, are equipotential be changed so as to make the new potential function between the plates a function $[V'=f(V)]$ of the old, the new inductivity must satisfy an equation of the form $\mu'=\Omega.\mu/f'(V)$. If the V and the μ surfaces are neither coincident nor orthogonal, V cannot be harmonic, and if V is given and one value of the inductivity found, no other value of the inductivity with the same level surfaces as this can be found except by altering the old value at every point in a constant ratio. If V does not satisfy Lamé's condition, a new value of the inductivity found by multiplying the old value by any point function orthogonal to V, will yield the same value of V, but the level surfaces of the inductivity will be altered. If the V and the μ surfaces are not coincident, no change of the inductivity which leaves its surfaces unchanged can make these surfaces equipotential.

If a mass of fluid, the characteristic equation of which is of the form $p=f(\rho, T)$, is at rest under the action of a conservative field of force the components of which are X, Y, Z,

$$\frac{\partial p}{\partial x}=\rho\cdot X, \quad \frac{\partial p}{\partial y}=\rho\cdot Y, \quad \frac{\partial p}{\partial z}=\rho\cdot Z. \tag{26}$$

It follows immediately from these equations that p and V must be colevel, and the normal derivative of p with respect to V shows that equilibrium is impossible unless the distribution of temperature is such that the equipotential surfaces are also isothermal.

If the scalar point function, W, is expressed in terms of the three orthogonal point functions, u, v, w, the square of the gradient of W is well known to be equal to

$$h_u^2\cdot\left(\frac{\partial W}{\partial u}\right)^2+h_v\cdot\left(\frac{\partial W}{\partial v}\right)^2+h_w^2\cdot\left(\frac{\partial W}{\partial w}\right)^2.$$

If the vector point function Q is expressed in terms of u, v, w, the divergence of Q is equal to

$$h_u\cdot h_v\cdot h_w\left[\frac{\partial}{\partial u}\left(\frac{Q_u}{h_v\cdot h_w}\right)+\frac{\partial}{\partial v}\left(\frac{Q_v}{h_u\cdot h_w}\right)+\frac{\partial}{\partial w}\left(\frac{Q_w}{h_u\cdot h_v}\right)\right].$$

If the normal derivatives of u and v with respect to w be denoted by $D_w u$ and $D_w v$, it follows from the definition that

$$D_w(u+v) = D_w u + D_w v, \qquad D_w u^n = n\cdot u^{n-1}\cdot D_w u,$$

$$D_w(u\cdot v) = v\cdot D_w u + u\cdot D_w v, \quad D_w\left(\frac{u}{v}\right) = \frac{v\cdot D_w u - u\cdot D_w v}{v^2},$$

$$D_w f(u) = f'(u)\cdot D_w(u).$$

The normal derivative of u with respect to v is a scalar function which, if differentiable, has a normal derivative with respect to v, and since by definition

$$D_v x = \frac{D_x v}{h_v^2}, \tag{27}$$

$$D_v h_v = \frac{1}{h_v^2}\left\{\frac{\partial h_v}{\partial x}\cdot\frac{\partial v}{\partial x} + \frac{\partial h_v}{\partial y}\cdot\frac{\partial v}{\partial y} + \frac{\partial h_v}{\partial z}\cdot\frac{\partial v}{\partial z}\right\}, \tag{28}$$

we may write

$$\begin{aligned} D_v^2 u = {} & \frac{1}{h_v^4}\left\{\frac{\partial^2 u}{\partial x^2}\cdot\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial^2 u}{\partial y^2}\cdot\left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial^2 u}{\partial z^2}\cdot\left(\frac{\partial v}{\partial z}\right)^2\right\} \\ & + \frac{2}{h_v^4}\left\{\frac{\partial^2 u}{\partial x\cdot\partial y}\cdot\frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y} + \frac{\partial^2 u}{\partial y\cdot\partial z}\cdot\frac{\partial v}{\partial y}\cdot\frac{\partial v}{\partial z} + \frac{\partial^2 u}{\partial z\cdot\partial x}\cdot\frac{\partial v}{\partial z}\cdot\frac{\partial v}{\partial x}\right\} \\ & + \frac{1}{h_v^3}\left\{\frac{\partial u}{\partial x}\left(\frac{\partial h_v}{\partial x} - 2\,\frac{\partial v}{\partial x}\cdot\frac{\partial h_v}{\partial v}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial h_v}{\partial y} - 2\frac{\partial v}{\partial y}\cdot\frac{\partial h_v}{\partial v}\right)\right. \\ & \left. + \frac{\partial u}{\partial z}\left(\frac{\partial h_v}{\partial z} - 2\frac{\partial v}{\partial z}\cdot\frac{\partial h_v}{\partial v}\right)\right\} \end{aligned} \tag{29}$$

$$\begin{aligned} D_w D_v u = {} & \frac{1}{h_v^2 h_w^2}\left\{\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial w}{\partial x}\cdot\frac{\partial v}{\partial x} + \frac{\partial^2 u}{\partial y^2}\cdot\frac{\partial w}{\partial y}\cdot\frac{\partial v}{\partial y} + \frac{\partial^2 u}{\partial z^2}\cdot\frac{\partial w}{\partial z}\cdot\frac{\partial v}{\partial z}\right\} \\ & + \frac{1}{h_v^2\cdot h_w^2}\left\{\frac{\partial^2 u}{\partial x\cdot\partial y}\left(\frac{\partial w}{\partial y}\cdot\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\cdot\frac{\partial v}{\partial y}\right) + \frac{\partial^2 u}{\partial y\cdot\partial z}\left(\frac{\partial w}{\partial z}\cdot\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\cdot\frac{\partial v}{\partial z}\right)\right. \\ & \left. + \frac{\partial^2 u}{\partial x\cdot\partial z}\left(\frac{\partial w}{\partial z}\cdot\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\cdot\frac{\partial v}{\partial z}\right)\right\} \\ & + \frac{1}{h_v^2\cdot h_w^2}\left\{\frac{\partial u}{\partial x}\left(\frac{\partial^2 v}{\partial x^2}\cdot\frac{\partial w}{\partial x} + \frac{\partial^2 v}{\partial y\cdot\partial x}\cdot\frac{\partial w}{\partial y} + \frac{\partial v}{\partial x\cdot\partial z}\cdot\frac{\partial w}{\partial z}\right)\right. \\ & \left. - \frac{2}{h_v}\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}\left(\frac{\partial h_v}{\partial x}\cdot\frac{\partial w}{\partial x} + \frac{\partial h_v}{\partial y}\cdot\frac{\partial w}{\partial y} + \frac{\partial h_v}{\partial z}\cdot\frac{\partial w}{\partial z}\right)\right\} \\ & \frac{1}{h_v^2\cdot h_w^2}\left\{\frac{\partial u}{\partial y}\left(\frac{\partial^2 v}{\partial y^2}\cdot\frac{\partial w}{\partial y} + \frac{\partial^2 v}{\partial z\cdot\partial y}\cdot\frac{\partial w}{\partial z} + \frac{\partial v}{\partial y\cdot\partial x}\cdot\frac{\partial w}{\partial y}\right)\right. \\ & \left. - \frac{2}{h_v}\cdot\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}\left(\frac{\partial h_v}{\partial y}\cdot\frac{\partial w}{\partial y} + \frac{\partial h_v}{\partial z}\cdot\frac{\partial w}{\partial z} + \frac{\partial h_v}{\partial x}\cdot\frac{\partial w}{\partial x}\right)\right\} \end{aligned}$$

$$\frac{1}{h_v^2 \cdot h_w^2}\left\{\frac{\partial u}{\partial z}\left(\frac{\partial^2 v}{\partial z^2}\cdot\frac{\partial w}{\partial z}+\frac{\partial^2 v}{\partial x\cdot\partial z}\cdot\frac{\partial w}{\partial x}+\frac{\partial v}{\partial z\cdot\partial y}\cdot\frac{\partial w}{\partial y}\right)\right.$$
$$\left.-\frac{2}{h_v}\cdot\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial z}\left(\frac{\partial h_v}{\partial z}\cdot\frac{\partial w}{\partial z}+\frac{\partial h_v}{\partial x}\cdot\frac{\partial w}{\partial x}+\frac{\partial h_v}{\partial y}\cdot\frac{\partial w}{\partial y}\right)\right\} \tag{30}$$

It is evident that $D_v D_w u$ is usually quite different from $D_w D_v u$.

In the transformation of a partial differential equation from one set of independent variables to another set which does not form an orthogonal system, derivatives occur which are not normal in the sense of the last paragraphs. If a mass of fluid is in motion under the action of given forces, it is usually convenient either to express the orthogonal coördinates of a particle which at the time t has the position (x, y, z) in terms of t and the coördinates x_0, y_0, z_0, which the same particle had at the origin of time, or to express x_0, y_0, z_0, as functions of x, y, z, t.

$$x_0 = f_1(x, y, z, t), \quad y_0 = f_2(x, y, z, t), \quad z_0 = f_3(x, y, z, t). \tag{31}$$

In this case, it frequently happens that the level surfaces of f_1, f_2, f_3, are not orthogonal. According as we use the "historical" or the "statistical" method of studying the motion, we shall express the pressure and the density in terms of x_0, y_0, z_0, t, or in terms of x, y, z, t. Suppose the second method to have been chosen, and $\partial p / \partial x$ to have been found by the aid of Euler's Equations of Motion and the Equation of Continuity, and suppose that $\partial p / \partial x_0$ is needed. We shall then have

$$\frac{\partial p}{\partial x_0} = \frac{\partial p}{\partial x}\cdot\frac{\partial x}{\partial x_0} + \frac{\partial p}{\partial y}\cdot\frac{\partial y}{\partial x_0} + \frac{\partial p}{\partial z}\cdot\frac{\partial z}{\partial x_0}. \tag{32}$$

If with the help of (31) we find the values of the determinants

$$L = \begin{vmatrix} \frac{\partial y_0}{\partial y} & \frac{\partial y_0}{\partial z} \\ \frac{\partial z_0}{\partial y} & \frac{\partial z_0}{\partial z} \end{vmatrix}, \quad M = \begin{vmatrix} \frac{\partial y_0}{\partial z} & \frac{\partial y_0}{\partial x} \\ \frac{\partial z_0}{\partial z} & \frac{\partial z_0}{\partial x} \end{vmatrix}, \quad N = \begin{vmatrix} \frac{\partial y_0}{\partial x} & \frac{\partial y_0}{\partial y} \\ \frac{\partial z_0}{\partial x} & \frac{\partial z_0}{\partial y} \end{vmatrix}, \tag{33}$$

and put

$$Q = L\cdot\frac{\partial x_0}{\partial x} + M\cdot\frac{\partial x_0}{\partial y} + N\cdot\frac{\partial x_0}{\partial z},$$

$$R^2 = L^2 + M^2 + N^2,$$

we may write the results of differentiating all the equations of (31) with respect to x_0, y_0, z_0, in the form

$$\frac{\partial x}{\partial x_0} = \frac{L}{Q}, \frac{\partial y}{\partial x_0} = \frac{M}{Q}, \frac{\partial z}{\partial x_0} = \frac{N}{Q}, \tag{34}$$

so that

$$\frac{\partial p}{\partial x_0} = \frac{\frac{L}{R}\cdot\frac{\partial p}{\partial x} + \frac{M}{R}\cdot\frac{\partial p}{\partial y} + \frac{N}{R}\cdot\frac{\partial p}{\partial z}}{\frac{L}{R}\cdot\frac{\partial x_0}{\partial x} + \frac{M}{R}\cdot\frac{\partial x_0}{\partial y} + \frac{N}{R}\cdot\frac{\partial x_0}{\partial z}}, \tag{35}$$

and this is evidently equal to (9), the ratio of the directional derivatives of p and x_0 taken in the direction (s) in the (x, y, z) space in which both y_0 and z_0 are constant. If (s, p), (s, x) represent the angles between s and the directions of the gradient vectors of p and x respectively,

$$\frac{\partial p}{\partial x_0} = \frac{h_p \cdot \cos(s, p)}{h_{x_0} \cdot \cos(s, x_0)}. \tag{36}$$

It is convenient, therefore, to define the derivative of a scalar point function, u, with respect to another scalar point function, v, at any given point in any direction (s), as the ratio of the directional derivatives of u and v taken at the point in the direction s.

Derivatives of this kind which frequently appear in two dimensional problems in Thermodynamics and in Hydrokinematics, usually involve, as has been said, a transformation from one set of coördinates to another which is not orthogonal.

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